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SHORT COMMUNICATION

A COMPARISON BETWEEN BOUNDARY FITTED COORDINATE SYSTEM AND FINITE ELEMENT METHOD IN SOLVING A HEAT CONDUCTION PROBLEM

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ABSTRACT

An attempt has been made to solve the heat conduction equation in a multiconnected domain using both boundary fitted coordinate system and finite element method. It has been found that boundary fitted coordinate system takes significantly less time in setting up the grid lines or mesh points compared to the finite element method of ANSYS. It has also been established that the former method takes much less time in obtaining a grid independent solution of the temperature field compared to the later one.

KEY WORDS Heat conduction problem BFC FEM ANSYS

INTRODUCTION

Analysis in the field of heat transfer has undergone radical change in methodology in recent years because of the growth in speed and power of modern digital computers. These changes fostered creation of various numerical techniques to solve the partial differential equation that governs conduction heat transfer. One technique, the finite difference method, approximates the differential equation by transforming it to a set of algebraic equations. The solution to the original problem is then the solution to the set of algebraic equations.

The heat conduction equation, or more generally the diffusion equation, is an elliptic type partial differential equation. Elliptic problems require information on all boundaries; therefore, the key to these problems is the accurate coupling of the boundaries of the interior. This is best accomplished when the boundaries coincide with the coordinate lines so that node points in the finite difference grid also coincide with the boundaries of the system under study. Finite difference expressions on or near boundaries, i.e., where boundaries and node points coincide, can be obtained without special interpolating formulas. However, many real problems involve irregular boundaries which require interpolation between boundaries and interior grid points. Such interpolations are inaccurate and produce large errors in the vicinity of strong curvature and shape discontinuities of the boundary. Alternatively the finite element method (FEM) is powerful in dealing with geometric complexity. However, automatic division into finite elements is no way a simple task and requires considerable experience. Although FEM can take care of the irregular geometry but specification of the Dirichlet and Neumann conditions on it is quite

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cumbersome while it is quite straight forward in boundary fitted coordinate system. Nevertheless FEM has emerged as a powerful tool in solving partial differential equations in irregular geometry. There are few packages available which do automatic mesh generation in finite element and solve quite complicated problems in both two and three dimensions. ANSYS is one[6] of them.

Here in this paper our aim is to solve the two-dimensional steady state heat conduction equation with arbitrary shape multiconnected domains in a rectangular plate using boundary fitted coordinate system and compare the accuracy of the result with that of finite element method of ANSYS. It is to be noted here that boundary conforming coordinate system can be generated much more easily by solving a set of Poisson equations than compared with its counterpart FEM mesh generator.

Methods for generation of curvilinear coordinate system, with all their details, are available in the literature[1–4]. Application of the method to the solution of the engineering problems is widely known. However, the comparison of the method with other available methods, so far its accuracy is concerned, is not much reported in literature. Goldman and Kao[5] have given a comparison of the method with the analytical solution for a very simple geometry with a circular hole in a square plate. For complex geometry like arbitrary shape multiconnected domains a direct comparison of BFC with finite element method, which is equally capable of handling the situation, is not available. The present work is geared towards this direction.

STATEMENT OF THE PROBLEM

The present work aims for the solution of a steady state temperature field in a square plate with two arbitrary shape holes in it (*Figure 1*) by utilizing the method of boundary fitted coordinates and tries to compare the accuracy of the result with the finite element method of ANSYS[6]. For this purpose a plate of side 10 units each is selected with the arbitrary shape holes in it as shown in *Figures 1* and *2*. Boundary fitted grid lines are obtained in the physical domain (*Figure 1*) by solving the following equations[1–3].

$$\alpha\frac{\partial^2 X}{\partial\xi^2} - 2\beta\frac{\partial^2 X}{\partial\xi\partial\eta} + \gamma\frac{\partial^2 X}{\partial\eta^2} + J^2P(\xi,\eta)\frac{\partial X}{\partial\xi} + J^2Q(\xi,\eta)\frac{\partial X}{\partial\eta} = 0 \tag{1a}$$

$$\alpha\frac{\partial^2 Y}{\partial\xi^2} - 2\beta\frac{\partial^2 Y}{\partial\xi\partial\eta} + \gamma\frac{\partial^2 Y}{\partial\eta^2} + J^2P(\xi,\eta)\frac{\partial Y}{\partial\xi} + J^2Q(\xi,\eta)\frac{\partial Y}{\partial\eta} = 0 \tag{1b}$$

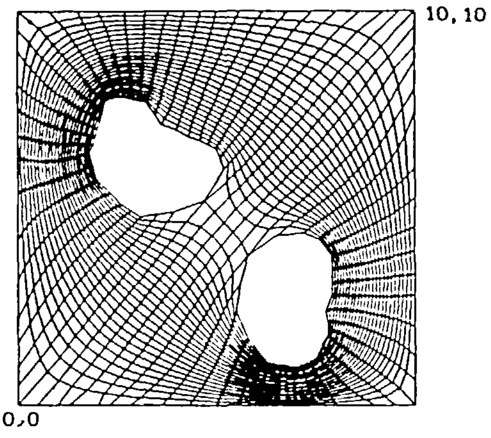

Figure 1 Boundary fitted coordinate lines in the physical plane

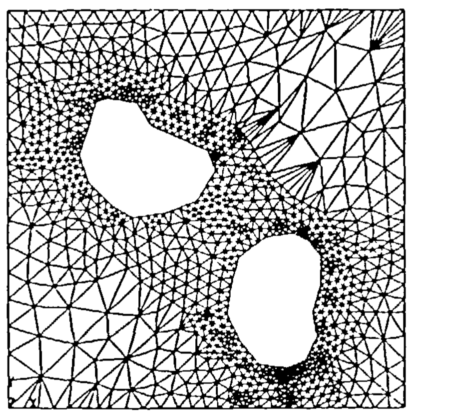

Figure 2 Finite element mesh generation in the physical plane (ANSYS)

where

$$\left.\begin{aligned}\alpha = X_\eta^2 + Y_\eta^2, \quad \beta = X_\xi X_\eta + Y_\xi Y_\eta \\ \gamma = X_\xi^2 + Y_\xi^2, \quad J = X_\xi Y_\eta - Y_\xi X_\eta\end{aligned}\right\} \tag{1c}$$

The subscripts ξ and η in (1c) denote first partial derivatives with respect to ξ and η. The quantity J is the Jacobian of transformation.

The steady state heat conduction equation applicable in the transformed plane may be expressed following the method of Uchikawa and Takeda[7] as:

$$\frac{\partial}{\partial \xi}\left(\frac{\alpha}{J}k\frac{\partial T}{\partial \xi} - \frac{\beta}{J}k\frac{\partial T}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\gamma}{J}k\frac{\partial T}{\partial \eta} - \frac{\beta}{J}k\frac{\partial T}{\partial \xi}\right) = 0 \tag{2}$$

where α, β, γ and J are stated in (1c). All of the coefficients α, β, etc., are calculated as a part of coordinate transformation and are known quantities. For the solution of (2) the following boundary conditions are employed in both the methods (FEM and BFC).

(a) $T = 1$ at $Y = 0$ (3a)

(b) $T = 1$ at $X = 10$ (3b)

(c) $\dfrac{\partial T}{\partial X} = 0$ at $X = 0$ (3c)

(d) $-k\dfrac{\partial T}{\partial Y} = \pm 0.2$ at $Y = 10$ (3d)
(heat flux going out or coming into the plate)

(e) the two arbitrary shape inner holes are kept at a constant temperature of $T = 0.2$.

Boundary conditions of the above form have been taken so as to include all types of boundary conditions in a single test problem.

METHOD OF SOLUTION

First, it is desirable to obtain a grid independent solution of the temperature field on the plate by finite element method. For this purpose triangular two-dimensional 6-node and quadrilateral 2D 4-node elements were tried with various mesh points subjected to the boundary conditions written in (3). To ascertain a grid independent solution the temperature field was plotted on several selected lines like, $X = 0, 2, 4, 6, 8$, and $Y = 2, 4, 6, 8$, and 10. Solution of the temperature field with 504 elements 1113 nodes, 1109 elements 2377 nodes, 1802 elements 3815 nodes (all triangular 2D) and 1610 elements 1710 nodes (quadrilateral 2D) was obtained and plotted on the selected lines. It was observed that solution with 1109 elements, 1610 elements and 1802 elements produced almost the same temperature field on all selected lines. So, as a final solution the temperature field with 1802 elements was selected for the purpose of comparison with BFC and the corresponding element plot is shown in *Figure 2*.

Regarding the solution of boundary fitted coordinate system, (1a) and (1b) were first expressed in finite difference form and solved by a point SOR (successive over relaxation) scheme with the coordinates of the physical domain as the boundary conditions in the transformed ξ–η plane. The coordinate control functions P and Q were set to zero on an experimental basis. The boundary fitted grid lines were obtained after the solution of (1) and are plotted in *Figure 1*. For the solution of (1) 65×25 (65 in ξ and 25 in η) mesh points were chosen with the relaxation factor as 1.8. It was observed that the solution of (1) with a SOR scheme requires relaxation parameter between 1.2 to 1.85.

The transformed energy equation (2) was expressed in a finite difference scheme with the ξ and η derivatives in central difference. The geometric parameters α/J, β/J *and* γ/J were calculated at the mid point of any two nodes. Thermal conductivity k, density ρ and specific heat c of the plate material were set to 1 for comparison purposes in both the calculations (present one and FEM), although (2) is a very convenient form for including variable property in the transformation process. The finite difference form of (2) was solved with point SOR scheme with the relaxation parameter varying between 1.4 to 1.8. The solution was assumed to have reached steady state when the maximum spatial variation of temperature over the whole domain was less than 10^{-7} from one iteration to the other. For the comparison of temperature field with FEM the temperature on the selected lines was plotted. It was found that mesh points of 58 × 20 and 65 × 25 could produce the same temperature profile as that of FEM on the selected lines (maximum deviation 0.7% from FEM with FEM as base) while mesh points of 40 × 25 and 48 × 22 produced maximum deviation of 6.2% and 3.4%, respectively, when plotted on the selected lines. So the solution with 65 × 25 mesh points was taken to be the grid independent solution for BFC.

RESULTS AND DISCUSSION

It is cited in the Introduction that boundary confirming coordinate system could be generated much more easily compared to its counterpart FEM mesh generator. The FEM mesh generator normally takes more time in approximately setting up the same number of node points compared to the BFC system. *Table 1* gives an indication of the CPU time (on an APOLLO DN 3500 machine) involved in setting up various number of mesh points in both the methods on the same physical domain (*Figures 1* and *2*). It is difficult to get identically the same number of node points in both the methods. But a rough estimation of the node point setting time can be obtained from *Table 1*. Clearly from *Table 1* it can be observed that BFC takes significantly less time in setting up almost equal number of node points as compared to FEM.

Table 1 CPU time involved in setting up mesh points on the physical domain (*Figures 1* and *2*) by FEM (ANSYS) and BFC

Method	Element type	No. of elements	No. of nodes	CPU time (sec)
FEM (ANSYS)	Triangular 2D-6-node	111	274	36.4
		504	1113	66.6
		1109	2377	130.4
		1802	3815	192.1
FEM (ANSYS)	Quadrilateral 4-node	68	81	25.5
		245	275	36.2
		532	569	55.5
		1410	1478	122.7
BFC	Quadrilateral 4-node	440	504	7.6
		550	616	9.4
		770	840	12.3
		990	1064	18.0
		1320	1400	27.1
		1536	1625	40.2
		1650	1736	42.4

It is well known that there will be grid clustering on the convex surfaces when the coordinate control function P and Q are set to zero. This is also evident from *Figure 1*. But in the FEM solution (*Figure 2*) mesh point density was deliberately kept high near the two holes in order to compare the isotherms with that of the solution of BFC. It can be observed from *Figure 3* that the isotherms obtained from both the methods are almost identical particularly near the zones of arbitrary shape holes. There is very little deviation in the upper left corner. It is suspected that this might have been caused because of the linear interpolations used in the contour plotting routine.

In *Figure 4* a comparison of the temperature profile (obtained using BFC) on the X wall ($X = 0$) and the Y wall ($Y = 10$) is made with the solution of FEM with the heat flux into the

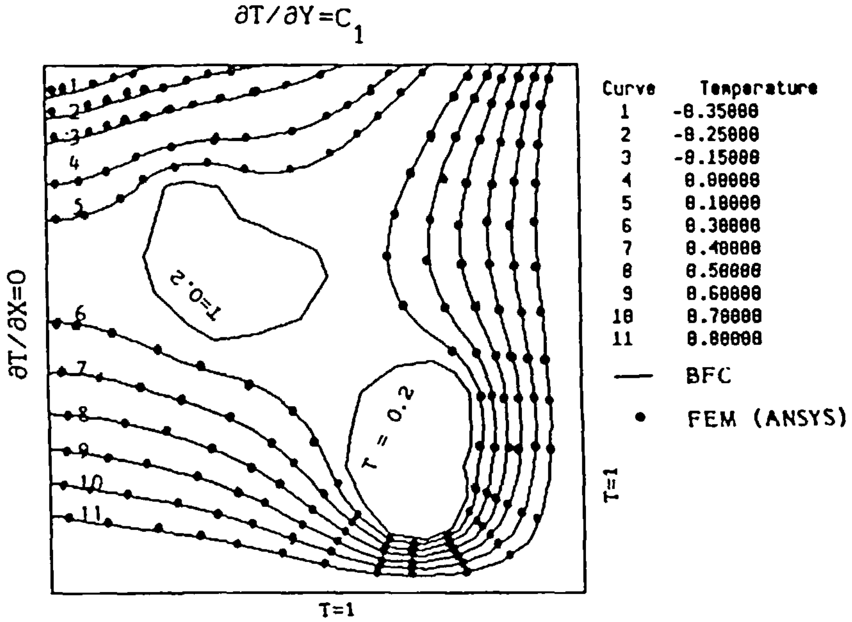

Figure 3 Comparison of isotherms at steady state with heat flux into the plate at 'Y' wall

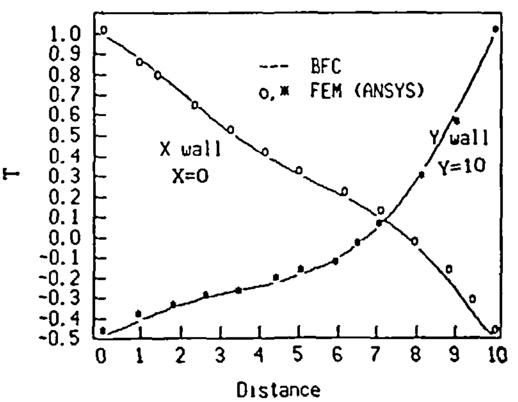

Figure 4 Steady state temperature distribution on the 'X' and 'Y' wall (heat flow in at 'Y' wall)

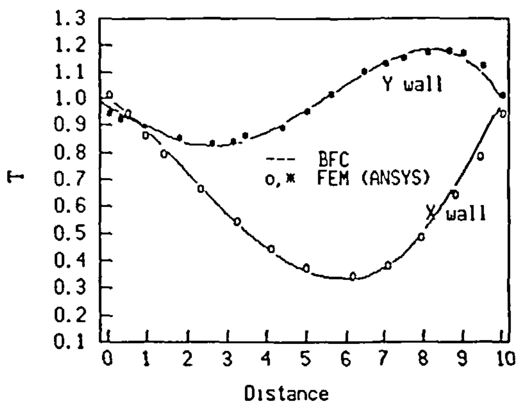

Figure 5 Steady state temperature distribution on the 'X' and 'Y' wall (heat flow out at 'Y' wall)

Table 2 CPU time required to obtain a grid independent solution of the temperature field in the physical domain (*Figure 1*)

Method	Element type	No. of elements	No. of nodes	CPU time (sec)
FEM	Triangular	1109	2377	209.1
(ANSYS)	2D-6-nodes	1802	3815	314.4
BFC	Quadrilateral 4-nodes	1536	1625	40.70

plate at the Y wall. The agreement seems to be quite satisfactory. Also in *Figure 5* a similar comparison has been made with the only change that the heat flux at the Y wall is going out of the plate. The temperature profiles on the two walls change drastically from those of *Figure 4* but still the solutions from the two different methods agree very well.

In the present calculation the heat flux is prescribed to be $\pm 0.2 = (-k\,\partial T/\partial Y)$ at $Y = 10$. But in the FEM package of ANSYS this gradient condition cannot be directly specified, instead the heat flow at each node may be prescribed. For this purpose the total heat flow across the wall ($Y = 10$) was calculated and was distributed proportionally among all the nodes on the line $Y = 10$ in the FEM calculation. While prescribing the inner holes at a temperature $T = 0.2$ all the nodes on the arbitrary boundary were to be identified manually and set to this value for the prediction of the temperature field using FEM. In case of BFC the arbitrary boundaries (two holes) lie on the straight line $\eta = 1$ and the line $Y = 10$ lies on line $\eta = 25$ making it very convenient to prescribe Dirichlet and Neumann boundary conditions on them. So a little discrepancy always remains between the two solvers while prescribing a gradient condition on the boundary. It is expected that some errors will always be brought in because of the above reason and it will be difficult to compare the relative accuracy of both the methods. However, judging from the view point of CPU time involved in both the methods in order to arrive at a grid independent solution (*Table 2*) it can be concluded that BFC has an advantage over FEM so far the solution of diffusion equation in a domain of arbitrary geometry is concerned.

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